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Lattice Coulomb gas representations of two-dimensional problems†

Leo P Kadanoff‡

Department of Physics and Materials Research Laboratory, Brown University, Providence, Rhode Island 02912, USA

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Abstract. Many of the standard two-dimensional problems of statistical physics can be transformed into ‘Coulomb gas’ problems in which there are two kinds of ‘charges’ represented by integers n and m . Such a transformation works for the Ising model, the three- and four-state Potts models, the Ashkin–Teller model, and many others. In general the n - n and m - m interactions have the Coulombic character in which the interaction is, for large separations, proportional to the logarithm of the distance. On the other hand, the $n(\mathbf{r})$ - $m(\mathbf{R})$ interaction is for large distances proportional to i times the angle $\Phi(\mathbf{r}-\mathbf{R})$ which measures the angular position of \mathbf{R} relative to \mathbf{r} . This latter interaction is akin to that between a magnetic monopole and an electric charge.

1. Introduction

1.1. Goals

In this paper, we shall reformulate many of the standard problems of statistical physics in terms of a pair of integer variables $n(\mathbf{r})$ and $m(\mathbf{R})$ which describe respectively quantum numbers for ‘charges’ and ‘magnetic monopoles’ confined to lattice sites \mathbf{r} and \mathbf{R} . In the recent literature there have appeared several formulations in which a single set of charges $n(\mathbf{r})$ interact via Coulomb-like potentials. (See, for example, Anderson and Yuval (1969), Kosterlitz and Thouless (1973), Kosterlitz (1974), and also the review of Kosterlitz and Thouless (1977).) However, in a recent publication Jose *et al* (1977) have stressed the usefulness of having two kinds of quantum numbers. (This approach was also apparently derived independently by Villain, whose work was described to me by Kosterlitz. For parallel discussions of the two-dimensional case and extension to higher dimensionality see Banks *et al* (1977) and Savit (1977).) These appear in an action (minus the Hamiltonian divided by kT) which takes the form

$$A[n, m] = \sum_{\mathbf{r}} Y(n(\mathbf{r})) + \sum_{\mathbf{R}} Y(m(\mathbf{R})) + A_0 + \frac{1}{2} \sum_{\mathbf{r}, \mathbf{r}'} n(\mathbf{r}) V_{n,n}(\mathbf{r}, \mathbf{r}') n(\mathbf{r}') + \frac{1}{2} \sum_{\mathbf{R}, \mathbf{R}'} m(\mathbf{R}) V_{m,m}(\mathbf{R}, \mathbf{R}') m(\mathbf{R}') + \sum_{\mathbf{r}, \mathbf{R}} m(\mathbf{R}) V_{m,n}(\mathbf{R}, \mathbf{r}) n(\mathbf{r}) \quad (1.1)$$

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‡ Address after September 1978: Department of Physics, University of Chicago, Chicago, Illinois 60637, USA.

when

$$\sum_{\mathbf{r}} n(\mathbf{r}) = 0 \quad \sum_{\mathbf{R}} m(\mathbf{R}) = 0. \quad (1.2)$$

Thus, the action (1.1) applies to the case in which the total electric and magnetic charges vanish. When they do not, the action is minus infinity. The statistical mechanics problem of calculating the partition function is then equivalent to doing the sum

$$Z = \sum_{\{n(\mathbf{r})\}} \sum_{\{m(\mathbf{R})\}} e^{A[n,m]} \quad (1.3)$$

with the sums restricted by the conditions (1.2).

The special character of the formulation (1.1) lies in the long-ranged nature of the interactions V . For large separations $V_{n,n}(\mathbf{r}, \mathbf{r}')$ and $V_{m,m}(\mathbf{r}, \mathbf{r}')$ are each proportional to $\ln |\mathbf{r} - \mathbf{r}'|$. This is, of course, the usual form of Coulomb interactions in two dimensions. The 'electric'-'magnetic' interaction $V_{m,n}(\mathbf{r}, \mathbf{r}')$ also has a long-ranged character. For large separations, it is proportional to $\Phi(\mathbf{r} - \mathbf{r}')$, where Φ is the usual angular variable defined by

$$\tan \Phi(\mathbf{r} - \mathbf{r}') = \frac{y - y'}{x - x'}. \quad (1.4)$$

As we will see in § 3 a large collection of the standard problems in two-dimensional statistical physics can be phrased in this manner, in which the difference between the problems are mostly reflected in changes in the parameters in the interactions, V . This unification is one of the advantages of the Coulombic formulation. Another advantage lies in the simplicity of the duality transformation (Kramers and Wannier 1941) in this language. The Kramers-Wannier duality is simply the replacement of n by m . If $V_{m,m} = V_{n,n}$, the problem is self-dual. As we shall see, many of the standard duality statements are made self-evident in this formulation.

There is an additional possible advantage of this formulation. In the usual handling of critical phenomena, long-ranged correlations are built up from short-ranged interactions. The interactions in (1.1) are, from the outset, long-ranged. Hence, one might be able to gain an insight into the nature of the long-ranged correlations via the action (1.1).

1.2. Methodology

Equations (1.1)–(1.3) define a description of our statistical mechanics problem which is, in some sense, analogous to a two-dimensional electrodynamics with charges and magnetic monopoles. We shall henceforth call this the 'electrodynamic' description of the problem. Our job is to connect this electrodynamic description to a 'standard' or Ising description of the very same problem. In this latter representation, the basic statistical variables are called $\sigma(\mathbf{r})$, which take on some pre-assigned set of values—for example $\sigma(\mathbf{r}) = \pm 1$. Then the action includes a contribution from all the sites:

$$A_{\text{site}}[\sigma] = \sum_{\mathbf{r}} h(\sigma(\mathbf{r})) \quad (1.5a)$$

and a set of interactions from nearest-neighbour bonds:

$$A_{\text{bond}}[\sigma] = \sum_{\langle \mathbf{r}, \mathbf{r}' \rangle} K(\sigma(\mathbf{r}) - \sigma(\mathbf{r}')). \quad (1.5b)$$

The partition function is then, of course, just the sum over all possible values of σ of the form

$$Z = \left(\prod_{\mathbf{r}} \sum_{\sigma(\mathbf{r})} \right) \exp (A_{\text{bond}}[\sigma] + A_{\text{site}}[\sigma]). \tag{1.6}$$

We wish to show how the very same partition function can be equivalently represented in the forms (1.3) and (1.4). To do this we go through an intermediate ‘gauge-theoretical’ representation of this problem. (References on ‘gauge theory’ models in statistical mechanics include Wegner (1971), Balian *et al* (1974, 1975a, b), Migdal (1975a, b), and Kadanoff (1977a, b). This third representation includes a set of variables and interactions defined on sites, on bonds, and on plaquettes. Each site contains a variable $\theta(\mathbf{r})$ which can range from $-\pi$ to π and a variable $n(\mathbf{r})$ which takes on integral values. The interaction on each site is of the form:

$$A_{\text{site}}[n, \theta] = \sum_{\mathbf{r}} Y(n(\mathbf{r})) + i\theta(\mathbf{r}) \cdot n(\mathbf{r}). \tag{1.7a}$$

Each nearest-neighbour bond contains a variable $l(\mathbf{r}, \mathbf{r}') = -l(\mathbf{r}', \mathbf{r})$ which takes on integral values. The interaction on the bond takes the form

$$A_{\text{bond}}[\theta, l] = - \sum_{(\mathbf{r}, \mathbf{r}')} \frac{2\pi}{2} \left(\frac{\theta(\mathbf{r}) - \theta(\mathbf{r}')}{2\pi} - l(\mathbf{r}, \mathbf{r}') \right) \cdot X \cdot \left(\frac{\theta(\mathbf{r}) - \theta(\mathbf{r}')}{2\pi} - l(\mathbf{r}, \mathbf{r}') \right). \tag{1.7b}$$

Finally, each square on the lattice forms a plaquette bounded by four bonds (see figure 1). On each plaquette we define a variable $m(\mathbf{R})$, \mathbf{R} being the centre of the plaquette, which is essentially the circulation of the l 's around the plaquette. That definition makes $m(\mathbf{R})$ an integer variable. The plaquette interaction is given by

$$\exp (A_{\text{plaquette}} [m, l]) = \prod_{\mathbf{R}} \delta_{m(\mathbf{R}), l_{12} + l_{23} + l_{34} + l_{41}} \exp (\tilde{Y}[m(\mathbf{R})]). \tag{1.7c}$$

In some cases we shall wish m, n, l and θ to have M components, hence the dots in equations (1.7a) and (1.7b) which indicate scalar products. In this situation the interaction constant will be an $M \times M$ matrix.

Our third description of the partition function will then be one in which Z is a sum over n, m, l and an integral over θ , i.e.

$$Z = \left(\prod_{\mathbf{r}} \int_{-\pi}^{\pi} \frac{d\theta(\mathbf{r})}{2\pi} \sum_{n(\mathbf{r})} \right) \left(\prod_{(\mathbf{r}, \mathbf{r}')} \sum_{l(\mathbf{r}, \mathbf{r}')} \right) \left(\prod_{\mathbf{R}} \sum_{m(\mathbf{R})} \right) e^{A[n, \theta, l, m]}. \tag{1.8}$$

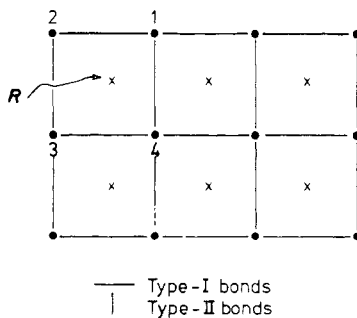


Figure 1. The square lattice. The full circles represent lattice sites, \mathbf{r} ; the crosses indicate sites of the dual lattice, \mathbf{R} . The definition of the plaquette variable shown is $m(\mathbf{R}) = l_{12} + l_{23} + l_{34} + l_{41}$.

Here the total action is, of course,

$$A[n, \theta, m, l] = A_{\text{site}}[n, \theta] + A_{\text{bond}}[\theta, l] + A_{\text{plaquette}}[m, l] \quad (1.9)$$

we call this third formulation of the action a gauge-theoretical one because $A[n, \theta, m, l]$ has an invariance under the replacement

$$\begin{aligned} \theta(\mathbf{r}) &\rightarrow \theta(\mathbf{r}) + 2\pi q(\mathbf{r}) \\ l(\mathbf{r}, \mathbf{r}') &\rightarrow l(\mathbf{r}, \mathbf{r}') + q(\mathbf{r}) - q(\mathbf{r}') \end{aligned} \quad (1.10)$$

for any set of integers $q(\mathbf{r})$.

We shall generate both the 'standard' description of the problem and the electrodynamic description by doing partial summations on the gauge-theoretical formulation. In particular, we shall show that the sum over the l 's and the θ 's yields an action of the form (1.1), i.e.

$$e^{A[n, m]} = \left(\prod_{(\mathbf{r}, \mathbf{r}')} \sum_{l(\mathbf{r}, \mathbf{r}')} \right) \left(\prod_{\mathbf{r}} \int_{-\pi}^{\pi} \frac{d\theta(\mathbf{r})}{2\pi} e^{A[n, \theta, l, m]} \right). \quad (1.11)$$

This correspondence between the electrodynamic and the gauge-theoretical formulations is developed in § 3.

The connection with the standard formulation of the problem is developed in § 2. Finally, in § 4 all the results are brought together and summarised.

2. Connections with standard problems

In this section, we reduce the gauge-theoretic formulation of our statistical mechanics problem to the standard formulation, which involves variables $\sigma(\mathbf{r})$ and nearest-neighbour interactions.

2.1. Elimination of $m(\mathbf{R})$ and $l(\mathbf{r}, \mathbf{r}')$

The first step in this reduction is to sum expression (1.8) over the variables $m(\mathbf{R})$ and $l(\mathbf{r}, \mathbf{r}')$. If one makes the specific choice

$$\tilde{Y}(m) = 0 \quad (2.1)$$

this sum is very simple indeed. We then find that

$$\left(\prod_{\mathbf{R}} \sum_{m(\mathbf{R})} \right) \exp(A_{\text{plaquette}}[m, l]) = 1. \quad (2.2)$$

The sums over l 's on different bonds are then independent of one another. We find that each bond has a coupling of the Villain (1975) form:

$$\exp[K(\theta(\mathbf{r}) - \theta(\mathbf{r}'))] = \sum_l \exp \left[-\pi \left(\frac{\theta(\mathbf{r}) - \theta(\mathbf{r}')}{2\pi} - l \right) \cdot \mathbf{X} \cdot \left(\frac{\theta(\mathbf{r}) - \theta(\mathbf{r}')}{2\pi} - l \right) \right]. \quad (2.3)$$

We wish to use equation (2.3) in three cases. In the first case the bonds are the usual nearest-neighbour bonds of the square lattice and θ , l , and \mathbf{X} are just simple real numbers. In the second case, we still have the same bond structure but θ and l are each vectors with M components while \mathbf{X} is a symmetrical $M \times M$ matrix. Then,

equation (2.3) will become

$$e^{K(\theta-\theta')} = \sum_{l_1, l_2, \dots, l_M} \exp \left[-\pi \sum_{j,k} \left(\frac{\theta_j - \theta'_j}{2\pi} - l_j \right) X_{jk} \left(\frac{\theta_k - \theta'_k}{2\pi} - l_k \right) \right]. \quad (2.4)$$

The Ashkin–Teller model (Ashkin and Teller 1943) can be represented by a situation in which $M = 2$. Then X_{jk} is a 2×2 matrix which we write as

$$\mathbf{X} = \begin{pmatrix} X_0 & X_1 \\ X_1 & X_0 \end{pmatrix} = X_0 + X_1 \tau_1. \quad (2.5)$$

Yet another two-component case arises when $\theta(\mathbf{r})$ has two components which appear on two different sublattices, each of which is a simple square lattice (see figure 2). We denote one sublattice by giving it the lattice sites \mathbf{r} and the other by giving it the site names \mathbf{R} . Then the basic coupling term indicated in figure 2 is $K(\theta_1(\mathbf{r}) - \theta_1(\mathbf{r}'), \theta_2(\mathbf{R}) - \theta_2(\mathbf{R}'))$ where the coupling function is identical to that given in equations (2.4) and (2.5). This is then the coupling structure appropriate for the eight-vertex model. (For references on the Ashkin–Teller and the eight-vertex models see Baxter (1971, 1972), Barber and Baxter (1973), Wegner (1972), Fan (1972), Fan and Wu (1970), Knops (1975), Wu (1977), Wu and Lin (1974), and Kadanoff and Wegner (1971).)

Now we have bond structures entirely similar to those of the standard problems.

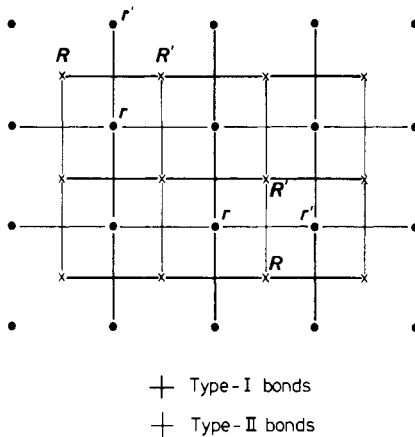


Figure 2. The lattice used for the eight-vertex model. The full circles represent lattice sites, \mathbf{r} ; the crosses indicate sites of the dual lattice, \mathbf{R} .

2.2. Going from $n(\mathbf{r})$ and $\theta(\mathbf{r})$ to $\sigma(\mathbf{r})$

In the gauge-theoretical formulation, we have at each site a summation over $n(\mathbf{r})$ and an integration over $\theta(\mathbf{r})$ of the form:

$$I = \sum_{n(\mathbf{r})} \int_{-\pi}^{\pi} \frac{d\theta(\mathbf{r})}{2\pi} \exp[in(\mathbf{r}) \cdot \theta(\mathbf{r}) + Y(n(\mathbf{r}))] f(\theta(\mathbf{r})). \quad (2.6)$$

Here $f(\theta)$ is some very complicated function determined by the integrations over all other variables.

We wish to reduce this integration to a discrete summation over some set of values of $\theta(\mathbf{r})$, i.e.

$$\theta(\mathbf{r}) = \sigma(\mathbf{r}) \frac{2\pi}{p} - \pi \quad (2.7)$$

with

$$\sigma(\mathbf{r}) = 1, 2, \dots, p. \quad (2.8)$$

This goal is easily accomplished with the aid of the Poisson sum formula which states

$$\sum_l e^{il\theta} = 2\pi \sum_k \delta(\theta - 2\pi k). \quad (2.9)$$

Then simply choose

$$e^{Y(n)} = \sum_{\sigma=1}^p \exp \left[-in \left(\frac{2\pi\sigma}{p} - \pi \right) + h(\sigma) \right] \quad (2.10)$$

and the expression (2.6) reduces to

$$I = \sum_{\sigma(\mathbf{r})=1}^p e^{h(\sigma(\mathbf{r}))} f \left(\frac{2\pi}{p} \sigma(\mathbf{r}) - \pi \right). \quad (2.11)$$

Here $h(\sigma(\mathbf{r}))$ is the site weight function of equation (1.5a).

In short, we have succeeded in converting the θ integrals to discrete sums and the bond interactions to simple nearest-neighbour couplings. In gross, the gauge problem has been reduced to the standard problem.

2.3. Examples

2.3.1. The Ising model. For the Ising case $\sigma = 1, 2$. Let $h(\sigma) = +h$ if $\sigma = 2$ and $-h$ if $\sigma = 1$. Then

$$e^{Y(n)} = \begin{cases} 2 \cosh h & \text{for } n \text{ even} \\ -2 \sinh h & \text{for } n \text{ odd.} \end{cases} \quad (2.12a)$$

If the magnetic field, h , is zero, then

$$e^{Y(n)} = \begin{cases} 2 & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd.} \end{cases} \quad (2.12b)$$

The bond interaction is

$$e^{K(\sigma-\sigma')} = \sum_l \exp \left[-\pi \left(\frac{\sigma-\sigma'}{2} - l \right)^2 X \right].$$

The standard way of writing this interaction is to take it to be $\exp(K_0 \pm K_1)$ depending upon whether σ is equal or unequal to σ' . Then

$$e^{K_0 + K_1} = \sum_l e^{-\pi X l^2} \quad (2.13)$$

$$e^{K_0 - K_1} = \sum_l e^{-\pi X (l - \frac{1}{2})^2}.$$

2.3.2. *The p-state model* The p -state (planar Potts) model is obtained with unit vertex weight if we choose

$$e^{Y(n)} = \begin{cases} p & \text{if } n/p = \text{an integer} \\ 0 & \text{otherwise.} \end{cases} \quad (2.14)$$

Then θ takes on the values (2.8) and the vertex function $h(\sigma)$ vanishes. In this case, the coupling on each bond takes the form

$$e^{K(\sigma(r)-\sigma(r'))} = \sum_l \exp \left[-\pi X \left(\frac{\sigma(r)-\sigma(r')}{p} - l \right)^2 \right]. \quad (2.15)$$

For $p = 3$ the coupling (2.15) is a representation of the three-state Potts model. For $p = 4$ and above, we have special cases of the planar Potts models.

2.3.3. *Ashkin–Teller model.* For the Ashkin–Teller model there are two variables $\sigma_1 = 1, 2$ and $\sigma_2 = 1, 2$ at each site. Accordingly we require a two-component θ variable and a two-component n variable. In order to achieve a vanishing on site interaction, we choose:

$$e^{Y(n_1, n_2)} = \begin{cases} 4 & \text{if } n_1 \text{ and } n_2 \text{ are even} \\ 0 & \text{otherwise.} \end{cases} \quad (2.16)$$

The inclusion of odd- n terms with a Y chosen so that $Y(n_1, n_2)$ is periodic under $n_1 \rightarrow n_1 + 2$ and/or $n_2 \rightarrow n_2 + 2$ will generate magnetic field terms coupled to σ_1, σ_2 , and $\sigma_1\sigma_2$.

The standard Ashkin–Teller interaction can be generated via equations (2.4) and (2.5), which yield a coupling structure

$$\begin{aligned} & \exp(K(\sigma_1 - \sigma'_1, \sigma_2 - \sigma'_2)) \\ &= \sum_{l_1, l_2} \exp \left\{ -\pi \left[\left(\frac{\sigma_1 - \sigma'_1}{2} - l_1 \right)^2 + (\sigma_2 - \sigma'_2 - l_2)^2 \right] X_0 \right. \\ & \quad \left. - 2 \left(\frac{\sigma_1 - \sigma'_1}{2} - l_1 \right) \left(\frac{\sigma_2 - \sigma'_2}{2} - l_2 \right) X_1 \right\}. \end{aligned} \quad (2.17)$$

The usual way of representing this coupling structure is to make use of the variables

$$S = 1 - (\sigma_1 - \sigma'_1)^2/2 \quad \tau = 1 - (\sigma_2 - \sigma'_2)^2/2 \quad (2.18)$$

which take on the values ± 1 . Then the coupling is written

$$\begin{aligned} & \exp[K_0 + K_1(S + \tau) + K_3S\tau] \\ &= \sum_{l_1, l_2} \exp \left\{ -\pi X_0 \left[\left(l_1 - \frac{1-S}{4} \right)^2 + \left(l_2 - \frac{1-\tau}{4} \right)^2 \right] \right. \\ & \quad \left. - 2\pi X_1 \left(l_1 - \frac{1-S}{4} \right) \left(l_2 - \frac{1-\tau}{4} \right) \right\}. \end{aligned} \quad (2.19)$$

Clearly X_0 represents a coupling within the set of variables σ_1 and within the set σ_2 while X_1 represent a σ_1 - σ_2 coupling.

Notice that the coupling (2.19) is unchanged under a change in sign of X_1 . (To see this replace l_2 by $-l_2 + (1-\tau)/2$.)

2.3.4. *The eight-vertex model.* The spin formulation of the eight-vertex model (Kadanoff and Wegner 1971, Fan and Wu 1970) is exactly the same as that of the Ashkin–Teller model except that the variables σ_1, θ_1, n_1 and σ_2, θ_2, n_2 appear on different sublattices. On each site, therefore, there is one n variable, one θ and one σ . If the magnetic field terms are zero, $Y(n)$ takes the form (2.12*b*). Equations (2.17) and (2.18) apply equally well to the eight-vertex model and the Ashkin–Teller model. However the standard notation for equation (2.19) is slightly different in that the left-hand side of this equation is often written

$$\exp[K_0 + K(S + \tau) + \lambda S\tau]. \tag{2.20}$$

(See Kadanoff and Wegner (1971).)

2.4. *Duality statements*

All the models mentioned here obey duality relations which take a simple form when the on-site fields, h , vanish. The duality statements are essentially that the partition functions of the models in question are left unchanged when the couplings are replaced by the coefficients in their Fourier series expansions. To be specific, equation (2.4) can be written in terms of the variables (2.8) as

$$e^{K(\sigma - \sigma')} = \sum_{l_1, \dots, l_M} \exp \left[-\pi \sum_{j,k} \left(\frac{\sigma_j - \sigma'_j}{p} - l_j \right) X_{jk} \left(\frac{\sigma_k - \sigma'_k}{p} - l_k \right) \right]. \tag{2.21}$$

This equation has a Fourier series representation:

$$e^{K(\sigma)} = \sum_{\mu_1=1}^p \sum_{\mu_2=1}^p \dots \sum_{\mu_M=1}^p \frac{1}{p^{M/2}} \exp \left(\tilde{K}(\mu) + 2\pi i \sum_k \frac{\sigma_k \mu_k}{p} \right). \tag{2.22}$$

The duality statement is that the partition function for the problem with couplings $K(\sigma)$ and that with couplings $\tilde{K}(\sigma)$ are identical.

Now calculate $\tilde{K}(\sigma)$. By inverting the Fourier transform we find

$$\begin{aligned} e^{K(\mu)} &= \frac{1}{p^{M/2}} \sum_{\sigma_1, \dots, \sigma_2} \exp(K(\sigma) - 2\pi i \sigma \cdot \mu / p) \\ &= \frac{1}{p^{M/2}} \sum_{\sigma=1}^p \sum_{l=-\infty}^{\infty} e^{-2\pi i \sigma \cdot \mu / p} \exp \left[-\frac{2\pi}{p} (\sigma - l) \cdot X \cdot \left(\frac{\sigma}{p} - l \right) \right]. \end{aligned}$$

Now the sum over l and the sum over σ can be combined into a single sum over a new variable $k = \sigma - pl$ to give

$$e^{K(\mu)} = \frac{1}{p^{M/2}} \sum_{k=-\infty}^{\infty} \exp \left(-2\pi i \frac{k \cdot \mu}{p} - \frac{2\pi}{p^2} k \cdot X \cdot k \right).$$

But the Poisson sum formula implies that

$$\sum_{k=-\infty}^{\infty} f(k) = \int_{-\infty}^{\infty} dk \sum_{l=-\infty}^{\infty} e^{2\pi i l k} f(k).$$

Thence, after a bit more calculation we find that

$$e^{K(\mu)} = \frac{(2\pi)^{M/2} p^{M/2}}{(\det X)^{1/2}} \sum_{l=-\infty}^{\infty} \exp \left[-\pi \left(\frac{\mu}{p} - l \right) \cdot p^2 X^{-1} \cdot \left(\frac{\mu}{p} - l \right) \right]. \tag{2.23}$$

Expression (2.23) for \tilde{K} is very similar indeed to expression (2.21) for K . Except for a change in prefactors, the coupling is exactly of the same form with X replaced according to

$$X \rightarrow \tilde{X} = p^2 X^{-1}. \tag{2.24}$$

For the Ising model, $p = 2$ and the duality transform is

$$X \rightarrow \tilde{X} = 4/X. \tag{2.25}$$

The model is self-dual at $X = 2$. This self-duality point is, in fact, the critical point of the Ising model!

For the Ashkin–Teller and Baxter models the duality statement is, once again, equation (2.25) but now X and \tilde{X} are 2×2 matrices. If we represent \mathbf{X} as before, in the form (2.5), then

$$\tilde{\mathbf{X}} = \frac{4(X_0 - X_1 \tau_1)}{X_0^2 - X_1^2}. \tag{2.26}$$

The model will be self-dual if

$$X_0^2 - X_1^2 = 4. \tag{2.27}$$

This is indeed a simple statement of the duality conditions for the Ashkin–Teller and the eight-vertex models.

3. To the electrodynamic representation

3.1. From gauge theory to a Gaussian integral

This section is directed toward the derivation of the electrodynamic representation of our problem starting from the gauge-theoretical representation. In particular, we seek to calculate

$$Z[n, m] = \left(\prod_{\mathbf{r}} \int_{-\pi}^{\pi} \frac{d\theta(\mathbf{r})}{2\pi} \right) \left(\prod_{(\mathbf{r}, \mathbf{r}')} \sum_{l(\mathbf{r}, \mathbf{r}')} \right) \exp(A_{\text{site}}[\theta, n] + A_{\text{bond}}[\theta, l] + A_{\text{plaquette}}[l, m]). \tag{3.1}$$

We would like to prove that $Z[n, m]$ is the exponential of the action described in equations (1.1). Since the terms $Y(n(\mathbf{r}))$ and $\tilde{Y}(m(\mathbf{R}))$ move through the sums and integrals in equation (3.1) quite without change, we can simplify our notation by simply ignoring these terms.

To evaluate the sums and integrals in equation (3.1), we shall make use of the gauge invariance of the theory, which is the statement that the integrand is invariant under the transformations (1.10):

$$\begin{aligned} \theta(\mathbf{r}) &\rightarrow \theta(\mathbf{r}) + 2\pi q(\mathbf{r}) = 2\pi\phi(\mathbf{r}) \\ l(\mathbf{r}, \mathbf{r}') &\rightarrow l(\mathbf{r}, \mathbf{r}') + q(\mathbf{r}) - q(\mathbf{r}') = l'(\mathbf{r}, \mathbf{r}') \end{aligned} \tag{3.2}$$

where the q 's are integers. Notice that our lattice shown in figure 1 can each be decomposed into two types of bonds. On the type-I bonds choose $q(\mathbf{r})$ so that $l'(\mathbf{r}, \mathbf{r}')$ vanishes. Then rewrite the sum (3.1) in terms of the ϕ 's and the l 's, which still exist

on the type-II bonds. The result is

$$\begin{aligned}
 Z(n, m) = & \prod_{\mathbf{r}} \left(\int_{-\pi}^{\pi} \frac{d\theta(\mathbf{r})}{2\pi} \sum_{q(n)} e^{in(\mathbf{r})\phi(\mathbf{r})} \right) \\
 & \times \left(\prod_{\text{type-I bonds}} \exp[-\pi(\phi(\mathbf{r}) - \phi(\mathbf{r}')) \cdot X \cdot (\phi(\mathbf{r}) - \phi(\mathbf{r}'))] \right) \\
 & \times \left(\prod_{\text{type-II bonds}} \sum_{l(\mathbf{r}, \mathbf{r}')} \exp[-\pi(\phi(\mathbf{r}) - \phi(\mathbf{r}') - l(\mathbf{r}, \mathbf{r}')) \cdot X \cdot (\phi(\mathbf{r}) - \phi(\mathbf{r}') - l(\mathbf{r}, \mathbf{r}'))] \right) \\
 & \times \prod_{\mathbf{R}} \delta_{l_{2,3} - l_{1,4}, m(\mathbf{R})}. \tag{3.3}
 \end{aligned}$$

Notice how the integrand in equation (3.3) does not depend upon $\theta(\mathbf{r})$ or $q(\mathbf{r})$ separately but only upon the combination $\phi = \theta/2\pi + q$. Thence we can convert the integral into Gaussian form by making the replacement

$$\int_{-\pi}^{\pi} \frac{d\theta(\mathbf{r})}{2\pi} \sum_{q(\mathbf{r})} \rightarrow \int_{-\infty}^{\infty} d\phi(\mathbf{r}). \tag{3.4}$$

If there are N sites on the \mathbf{r} lattice equation (3.3) contains N summations over l' and N delta symbols which define $l'(\mathbf{r}, \mathbf{r}')$ in terms of the $m(\mathbf{R})$. Thus one can solve for the l' 's in terms of the m 's in the form

$$l'(\mathbf{r}, \mathbf{r}') = l(\mathbf{r}, \mathbf{r}', [m]) = - \sum_{k=0}^{\infty} m \left[\left(\frac{\mathbf{r} + \mathbf{r}'}{2} \right) + (k + \frac{1}{2}) \hat{e}_1 \right]. \tag{3.5}$$

Here \hat{e}_1 is a unit vector in the x direction and \mathbf{r} lies just above \mathbf{r}' .

At this point, equation (3.3) reduces to just a Gaussian integral:

$$Z[n, m] = \left(\prod_{\mathbf{r}} \int d\phi(\mathbf{r}) \right) e^Q$$

where

$$\begin{aligned}
 Q = & \sum_{\mathbf{r}} in(\mathbf{r})\phi(\mathbf{r})2\pi - \sum_{\text{type-I bonds}} \pi(\phi(\mathbf{r}) - \phi(\mathbf{r}')) \cdot X \cdot (\phi(\mathbf{r}) - \phi(\mathbf{r}')) \\
 & - \sum_{\text{type-II bonds}} \pi(\phi(\mathbf{r}) - \phi(\mathbf{r}') - l(\mathbf{r}, \mathbf{r}'; [m])) \cdot X \cdot (\phi(\mathbf{r}) - \phi(\mathbf{r}') - l(\mathbf{r}, \mathbf{r}'; [m])). \tag{3.6}
 \end{aligned}$$

Such a Gaussian integral can of course be evaluated immediately. In general the evaluation of a Gaussian integral is given by

$$I = \left(\prod_i \int_{-\infty}^{\infty} d\phi_i \right) \exp \left(-\frac{1}{2} \sum_{i,j} \phi_i G_{ij}^{-1} \phi_j + \sum_i \alpha_i \phi_i \right) = [\det(2\pi G)]^{1/2} \exp \left(\frac{1}{2} \sum_{i,j} \alpha_i G_{ij} \alpha_j \right) \tag{3.7}$$

where \mathbf{G} is the matrix inverse of \mathbf{G}^{-1} . Expression (3.6) has a form very similar to that of (3.7), except that there are extra quadratic terms in $l[m]$. Hence we conclude that the result of doing the integral is that $\ln Z[n, m]$ takes the form of a constant plus a quadratic form in n and m . In this way, we see that expression (1.1) is indeed justified.

Next we must turn to evaluating the parameters A_0 , $V_{n,n}$, $V_{m,m}$, $V_{n,m}$ and $V_{m,n}$ in that expression.

3.2. The Green function

Expression (3.6) has the ϕ 's coupled together by an inverse Green function of the form

$$G_{jk}^{-1}(\mathbf{r}, \mathbf{r}') = 2\pi X_{jk} \left(4\delta_{\mathbf{r},\mathbf{r}'} - \sum_{\hat{\epsilon}} \delta_{\mathbf{r},\mathbf{r}'+\hat{\epsilon}} \right) \tag{3.8}$$

where the $\hat{\epsilon}$ are the nearest-neighbour vectors on the lattice. Fourier transformation enables us to calculate the Green function immediately. We write the result as

$$G(\mathbf{r}, \mathbf{r}') = G(0, 0) - \frac{1}{(2\pi)^2} X^{-1} V_R(\mathbf{r}, \mathbf{r}'). \tag{3.9}$$

Here $G(0, 0)$ is the infinite constant

$$G(0, 0) = \frac{X^{-1}}{2\pi} \int_{-\pi}^{\pi} \frac{dq_x}{2\pi} \int_{-\pi}^{\pi} \frac{dq_y}{2\pi} \frac{1}{4 - 2 \cos q_x - 2 \cos q_y} \tag{3.10}$$

while $V(\mathbf{r}, \mathbf{r}')$ is the finite quantity

$$V_R(\mathbf{r} - \mathbf{r}') = \int_{-\pi}^{\pi} \frac{dq_x}{2\pi} \int_{-\pi}^{\pi} dq_y \frac{1 - e^{i\mathbf{q} \cdot (\mathbf{r} - \mathbf{r}')}}{4 - 2 \cos q_x - 2 \cos q_y}. \tag{3.11}$$

Notice that for large separations, $|\mathbf{r} - \mathbf{r}'|$,

$$V_R(\mathbf{r}, \mathbf{r}') \rightarrow \ln |\mathbf{r} - \mathbf{r}'|. \tag{3.12}$$

In our later analysis, we shall need another function $V_I(\mathbf{R}, \mathbf{r})$ to describe the $n-m$ couplings. Here \mathbf{R} is a site on the dual lattice and V_I is defined in terms of $V_R(\mathbf{r}, \mathbf{r}') = V_R(x - x', y - y')$ by

$$\begin{aligned} V_I(\mathbf{R} - \mathbf{r}) &= \lim_{\epsilon \rightarrow 0^+} \sum_{j=0}^{\infty} \epsilon^{-j} (V_R(X - x - \frac{1}{2} - j, Y - \frac{1}{2} - y) - V_R(X - x - \frac{1}{2} - j, Y + \frac{1}{2} - y)) \\ &= \lim_{\epsilon \rightarrow 0^+} \int_{-\pi}^{\pi} \frac{dq_x}{2\pi} \int_{-\pi}^{\pi} dq_y \frac{e^{i\mathbf{q} \cdot (\mathbf{R} - \mathbf{r})}}{4 - 2 \cos q_x - 2 \cos q_y} \frac{(-2i) \sin(q_y/2)}{e^{-\epsilon} e^{iq_x/2} - e^{-iq_x/2}}. \end{aligned} \tag{3.13}$$

The form of $V_I(\mathbf{R} - \mathbf{r})$ is relatively simple. For large values of the separation $\mathbf{R} - \mathbf{r}$ we have

$$\frac{\partial V_I}{\partial x}(\mathbf{r}) = -\frac{\partial V_R}{\partial y}(\mathbf{r}) = -\frac{y}{x^2 + y^2}. \tag{3.14}$$

This expression is true except when \mathbf{r} lies on the negative x axis. On that axis V_I has a discontinuity of strength 2π . Furthermore, $V_I(\mathbf{r})$ changes sign when $y \rightarrow -y$. All these properties together imply that $-V_I(\mathbf{r})$ is, for large \mathbf{r} , just the angle $\Phi(\mathbf{r})$ which obeys

$$\tan \Phi(\mathbf{r}) = y/x \tag{3.15}$$

with the conditions that $\Phi(\mathbf{r}) \rightarrow 0$ as $y \rightarrow 0$ when x is positive and that $\Phi(\mathbf{r})$ jumps from π to $-\pi$ as y passes through zero from above with x negative. So in the asymptotic limit

$$V_I(\mathbf{R} - \mathbf{r}) \rightarrow -\Phi(\mathbf{R} - \mathbf{r}). \tag{3.16}$$

Actually this asymptotic form even holds quite well for small values of $\mathbf{R} - \mathbf{r}$. When $\mathbf{r} = (\frac{1}{2}, \frac{1}{2}), (-\frac{1}{2}, \frac{1}{2}), (-\frac{1}{2}, -\frac{1}{2}), (\frac{1}{2}, -\frac{1}{2})$, $V_1(\mathbf{r})$ equals respectively $-\pi/4, -3\pi/4, 3\pi/4$, and $\pi/4$ —just as one would get from the asymptotic expression.

3.3. Coupling terms

With this Green function evaluated, one can start doing the integral (3.6). When all the m 's are zero, one finds by applying (3.9) that

$$\begin{aligned} \ln Z[n, 0] = & -\frac{1}{2} \sum_{\mathbf{r}} \left(\ln \det X + \int \frac{dq}{(2\pi)^2} \ln(4 - 2 \cos q_x - 2 \cos q_y) \right) \\ & - \frac{1}{2} \sum_{j,k} \left(\sum_{\mathbf{r}} n_j(\mathbf{r}) \right) (2\pi)^2 G_{jk}(\mathbf{0}, \mathbf{0}) \left(\sum_{\mathbf{r}'} n_k(\mathbf{r}') \right) \\ & + \frac{1}{2} \sum_{j,k} \sum_{\mathbf{r}, \mathbf{r}'} n_j(\mathbf{r}) (X^{-1})_{jk} V_{\mathbf{R}}(\mathbf{r} - \mathbf{r}') n_k(\mathbf{r}'). \end{aligned} \tag{3.17}$$

The first term in equation (3.17) is the constant term A_0 in equation (1.1). Since $G_{jk}(\mathbf{0}, \mathbf{0}) = \infty$, the second term reduces $Z[n, m]$ to zero unless the first of conditions (1.2), i.e.

$$\sum_{\mathbf{r}} n_j(\mathbf{r}) = 0,$$

is satisfied. The third term in equation (3.17) establishes the value of $V_{n,n}(\mathbf{r}, \mathbf{r}')$ to be

$$V_{n,n}(\mathbf{r}, \mathbf{r}') = (X^{-1})_{jk} V_{\mathbf{R}}(\mathbf{r} - \mathbf{r}'). \tag{3.18}$$

The next step is to consider the terms bilinear in n and l which emerge from equation (3.8). These terms in $\ln Z[n, m]$ are

$$\begin{aligned} & 2 \sum_{\mathbf{R}, \mathbf{r}} m(\mathbf{R}) \cdot V_{m,n}(\mathbf{R}, \mathbf{r}) \cdot n(\mathbf{r}) \\ & = \sum_{\langle \mathbf{r}_1, \mathbf{r}_2 \rangle} \sum_{\mathbf{r}} \sum_{j,k,p} i l_j(\mathbf{r}_1, \mathbf{r}_2; [m]) X_{jk} (C_{kp}(\mathbf{r}_1, \mathbf{r}) - G_{kp}(\mathbf{r}_2, \mathbf{r})) n_p(\mathbf{r}). \end{aligned}$$

A substitution of the expression for l in terms of m (equation (3.6)) immediately gives an evaluation of the next coupling term as

$$V_{m,n}(\mathbf{R}, \mathbf{r}) = i \delta_{j,k} V_1(\mathbf{R} - \mathbf{r}). \tag{3.19}$$

The final term in $\ln Z[n, m]$ is bilinear in $l(\mathbf{r}, \mathbf{r}')$ and hence bilinear in $m(\mathbf{r}, \mathbf{r}')$. This term has the form

$$\begin{aligned} & -\pi \sum_{\langle \mathbf{r}, \mathbf{r}' \rangle} l(\mathbf{r}, \mathbf{r}') \cdot X \cdot l(\mathbf{r}, \mathbf{r}') + \frac{1}{2} \sum_{\langle \mathbf{r}_1, \mathbf{r}'_1 \rangle} \sum_{\langle \mathbf{r}_2, \mathbf{r}'_2 \rangle} l(\mathbf{r}_1, \mathbf{r}'_1) \cdot X \cdot (2\pi)^2 \\ & \quad \times [G(\mathbf{r}_1, \mathbf{r}_2) + G(\mathbf{r}'_1, \mathbf{r}'_2) - G(\mathbf{r}_1, \mathbf{r}'_2) - G(\mathbf{r}_2, \mathbf{r}'_1)] \cdot l(\mathbf{r}_2, \mathbf{r}'_2). \end{aligned} \tag{3.20}$$

Here the sums are over type-II bonds. This term may be rewritten as a sum over m 's in the form

$$-\frac{1}{2} \sum_{\mathbf{R}, \mathbf{R}'} m(\mathbf{R}) \cdot F(\mathbf{F}, \mathbf{R}') m(\mathbf{R}') \tag{3.21}$$

with

$$F(\mathbf{R}, \mathbf{R}') = 2\pi X \delta_{\mathbf{e}_2, \mathbf{R} - \mathbf{R}', 0} |(\mathbf{R} - \mathbf{R}') \cdot \hat{\mathbf{e}}_1| - (2\pi)^2 \sum_{l=0}^{\infty} \sum_{l'=0}^{\infty} X \cdot \{ \quad \} \cdot X \tag{3.22a}$$

and

$$\{ \} = \sum_{\sigma=\pm 1} \sum_{\sigma'=\pm 1} \sigma\sigma' G(\mathbf{R} - l\hat{e}_1 - \frac{1}{2}(\hat{e}_1 + \sigma\hat{e}_2)), \quad \mathbf{R}' = l'\hat{e}_1 - \frac{1}{2}(\hat{e}_1 + \sigma'\hat{e}_2). \quad (3.22b)$$

But notice that $F(\mathbf{R}, \mathbf{R}')$ obeys

$$4F(\mathbf{F}, \mathbf{r}') - \sum_{\hat{e}} F(\mathbf{R} + \hat{e}, \mathbf{R}') = \delta_{\mathbf{R}, \mathbf{R}'} 2\pi X.$$

Therefore $F(\mathbf{R}, \mathbf{R}')$ is $X \cdot G(\mathbf{R}, \mathbf{R}') \cdot X$, where G is the Green function which we defined in equation (3.10).

In summary, this final term in $\ln Z[n, m]$ is

$$\ln Z[0, m] = \dots - \frac{1}{2} \sum_{\mathbf{R}, \mathbf{R}'} m(\mathbf{R}) \cdot X \cdot G(\mathbf{R}, \mathbf{R}') \cdot X \cdot m(\mathbf{R}'). \quad (3.23)$$

Without any further work we can conclude that $Z[n, m]$ vanishes unless

$$\sum_{\mathbf{R}} m_k(\mathbf{R}) = 0 \quad (3.24)$$

and that

$$V_{m,m}(\mathbf{R}, \mathbf{R}') = X_{jk} V_{\mathbf{R}}(\mathbf{R} - \mathbf{R}'). \quad (3.25)$$

Thus all terms in equation (1.1) have been evaluated.

3.4. Another lattice

To show how the eight-vertex model fits into this picture, we make use of the two interpenetrating square lattices shown in figure 2. Each lattice site of the first lattice, labeled by \mathbf{r} , contains variables $\theta_1(\mathbf{r})$, $n_1(\mathbf{r})$ and $m_2(\mathbf{r})$. The other lattice has sites labelled by \mathbf{R} and has on each site variables $\theta_2(\mathbf{R})$, $n_2(\mathbf{R})$ and $m_1(\mathbf{R})$. Notice that there are two types of bonds indicated in figure 2. Both bonds have a coupling structure

$$- \pi \psi \cdot \mathbf{X} \cdot \psi \quad (3.26)$$

where \mathbf{X} is a 2×2 matrix given by equation (2.5). For the type-II bond,

$$\begin{aligned} \psi_1 &= \frac{\theta_1(\mathbf{r} + \hat{e}_1) - \theta_1(\mathbf{r})}{2\pi} - l_1(\mathbf{r} + \hat{e}_1, \mathbf{r}) \\ \psi_2 &= \frac{\theta_2(\mathbf{R} + \hat{e}_2) - \theta_2(\mathbf{R})}{2\pi} - l_2(\mathbf{R} + \hat{e}_2, \mathbf{R}). \end{aligned} \quad (3.27a)$$

On the other hand, for the type-I bond, we choose

$$\begin{aligned} \psi_1 &= \frac{\theta_1(\mathbf{r} + \hat{e}_2) - \theta_1(\mathbf{r})}{2\pi} - l_1(\mathbf{r} + \hat{e}_2, \mathbf{r}) \\ \psi_2 &= \frac{\theta_2(\mathbf{R}) - \theta_2(\mathbf{R} + \hat{e}_1)}{2\pi} - l_2(\mathbf{R}, \mathbf{R} + \hat{e}_1). \end{aligned} \quad (3.27b)$$

Here, \hat{e}_1 and \hat{e}_2 are unit vectors in the x and y directions respectively.

Now we are ready to repeat the calculation which led from equation (3.1) to equation (3.25). Start by choosing a gauge in which $l_1(\mathbf{r}, \mathbf{r}')$ vanishes on type-II and $l_2(\mathbf{r}, \mathbf{r}')$ vanishes on type-I bonds. Then, for both types of bonds, in this gauge,

$$l_k(\mathbf{r} + \hat{\mathbf{e}}_2, \mathbf{r}) = - \sum_{l=0}^{\infty} m_k(\mathbf{r} + \hat{\mathbf{e}}_2/2 + (l + \frac{1}{2}) \hat{\mathbf{e}}_1). \tag{3.28}$$

Rewrite equation (3.6) for this new case. Denote by s a position variable which can take on either \mathbf{r} - or \mathbf{R} -values. Let, for example, $n(s)$ be $n_1(\mathbf{r})$ when s is on the first sublattice and $n_2(\mathbf{R})$ when s is on the second. With this convention

$$Z[n, m] = \prod_s \int d\mathbf{e}(s) e^{Q_0 + Q_1}$$

$$Q_0 = \sum_s 2\pi i n(s) \phi(s) - \pi X_0 \sum_s [(\phi(s) - \phi(s + \hat{\mathbf{e}}_1))^2 + (\phi(s) - \phi(s + \hat{\mathbf{e}}_2))^2 + (l(s + \hat{\mathbf{e}}_2, s))^2]$$

$$Q_1 = 2\pi \sum_{\mathbf{r}} l_1(\mathbf{r} + \hat{\mathbf{e}}_2, \mathbf{r}) [X_0(\phi_1(\mathbf{r} + \hat{\mathbf{e}}_2) - \phi_1(\mathbf{r})) - X_1(\phi_2(\mathbf{r} + \frac{1}{2}\hat{\mathbf{e}}_2 + \frac{1}{2}\hat{\mathbf{e}}_1) - \phi_2(\mathbf{r} + \frac{1}{2}\hat{\mathbf{e}}_2 - \frac{1}{2}\hat{\mathbf{e}}_1))] + 2\pi \sum_{\mathbf{R}} l_2(\mathbf{R} + \hat{\mathbf{e}}_2, \mathbf{R}) \times [X_0(\phi_2(\mathbf{R} + \hat{\mathbf{e}}_2) - \phi_2(\mathbf{R})) + X_1(\phi_1(\mathbf{R} + \frac{1}{2}\hat{\mathbf{e}}_1 + \frac{1}{2}\hat{\mathbf{e}}_2) - \phi_1(\mathbf{R} + \frac{1}{2}\hat{\mathbf{e}}_2 - \frac{1}{2}\hat{\mathbf{e}}_1))]. \tag{3.29}$$

Repeat the previous analysis. Then one finds, once more, a coupling with a structure given by equations (1.1) with terms

$$A_0 = - \sum_{\mathbf{r}} \int \frac{dq_x}{2\pi} \frac{dq_y}{2\pi} \ln [X_0(4 - 2 \cos q_x - 2 \cos q_y)] \tag{3.30}$$

an n - n coupling

$$V_{n,n}(s, s') = \delta_{j,k} X_0^{-1} V_{\mathbf{R}}(s - s') \tag{3.31}$$

an m - m coupling

$$V_{m,m}(s, s') = \delta_{i,k} \frac{X_0^2 - X_1^2}{X_0} V_{\mathbf{R}}(s - s')$$

and an n - m coupling

$$V_{n,m}(s, s') = \begin{bmatrix} V_{\mathbf{I}}(s - s') i & \frac{X_1}{X_0} V_{\mathbf{R}}(s - s') \\ \frac{X_1}{X_0} V_{\mathbf{R}}(s - s') & i V_{\mathbf{I}}(s - s') \end{bmatrix} \tag{3.32}$$

4. Representations for specific systems

In this section, we summarise the results of the previous sections by describing how the electrodynamic representations work out for the standard problems described in § 2.

4.1. The p -state case

Let $\sigma(\mathbf{r})$ take on integer values between 1 and p as described in equation (2.8a). Let the coupling between nearest-neighbour σ 's be that described by equation (2.15). Take the vertex function $h(\sigma)$ to be zero so that there is no breaking of the symmetry under the displacement of $\sigma(\mathbf{r})$ by an integer, q , i.e.

$$\sigma(\mathbf{r}) \rightarrow \sigma(\mathbf{r}) + q.$$

According to equation (2.14) this latter condition forces $n(\mathbf{r})$ to be p times an integer. For this reason we define variables which take on all integer values:

$$N(\mathbf{r}) = n(\mathbf{r})/p \quad M(\mathbf{R}) = m(\mathbf{R}) \tag{4.1}$$

and say that the p -state model has a partition function which is given by equations (1.1)–(1.3), i.e. a sum over all integral values of N and M :

$$Z = \sum_{\{M, N\}} e^{A[N, M]} \tag{4.2a}$$

with

$$A[N, M] = -\infty, \quad \text{if } \sum_{\mathbf{r}} N(\mathbf{r}) \neq 0 \text{ or } \sum_{\mathbf{R}} M(\mathbf{R}) \neq 0 \tag{4.2b}$$

and, otherwise,

$$A[N, M] = V[N, M] + A_0. \tag{4.2c}$$

In this case

$$V[N, M] = \frac{1}{2} \sum_{\mathbf{r}, \mathbf{r}'} N(\mathbf{r}) V_{\mathbf{R}}(\mathbf{r} - \mathbf{r}') N(\mathbf{r}') p^2 / X + i \sum_{\mathbf{r}, \mathbf{R}} M(\mathbf{R}) V_{\mathbf{I}}(\mathbf{R} - \mathbf{r}) N(\mathbf{r}) p + \frac{1}{2} \sum_{\mathbf{R}, \mathbf{R}'} M(\mathbf{R}) V_{\mathbf{R}}(\mathbf{R} - \mathbf{R}') M(\mathbf{R}') X. \tag{4.3}$$

In equation (2.24), we mentioned that the p -state model had a dual symmetry in which the partition function had a very simple transformation under the replacement of the coupling X by $\tilde{X} = p^2/X$. This symmetry is manifestly demonstrated in equation (4.3) as a symmetry of the entire problem under

$$N \leftrightarrow M \quad X \leftrightarrow p^2/X. \tag{4.4}$$

Therefore if the partition function has a singularity at some value of X , say X_c , then it also has a singularity at

$$\tilde{X}_c = p^2/X_c.$$

For $p = 2$ and 3, there is but one singularity at the self-dual point for which $X_c = \tilde{X}_c$ or

$$X = X^* = p. \tag{4.5}$$

4.2. Ashkin–Teller and eight-vertex models

For the Ashkin–Teller model, n and m have two components and $p = 2$. Thus, we define $N(\mathbf{r})$ and $M(\mathbf{R})$ by

$$N_k(\mathbf{r}) = n_k(\mathbf{r})/2 \quad M_k(\mathbf{R}) = m_k(\mathbf{R}). \tag{4.6}$$

Here \mathbf{X} is a 2×2 matrix of the form

$$\mathbf{X} = X_0 + X_1 \tau_1$$

so that, in this case, instead of equation (4.3), we find

$$\begin{aligned} V[N, M] = & \frac{1}{2} \sum_{r, r'} N(r) \cdot V_R(r - r')(X_0 - X_1 \tau_1) N(r') \frac{4}{X_0^2 - X_1^2} \\ & + 2i \sum_{r, \mathbf{R}, k} N_k(r) V_1(r - \mathbf{R}) M_k(\mathbf{R}) \\ & + \frac{1}{2} \sum_{\mathbf{R}, \mathbf{R}'} M(\mathbf{R}) \cdot V_R(\mathbf{R} - \mathbf{R}')(X_0 + X_1 \tau_1) \cdot M(\mathbf{R}'). \end{aligned} \tag{4.7}$$

The duality transformation for this problem is

$$\begin{aligned} N_1(r) &\leftrightarrow M_1(\mathbf{R}) & N_2(r) &\leftrightarrow -M_2(\mathbf{R}) \\ X_0 &\leftrightarrow 4X_0 / (X_0^2 - X_1^2) & X_1 &\leftrightarrow 4X_1 / (X_0^2 - X_1^2). \end{aligned} \tag{4.8}$$

For real values of X_1 in the region $X_1^2 < X_0^2$ the problem has a critical line in which the behaviour is a function of X_1 . This line occurs at critical values defined by

$$(X_0^*)^2 - (X_1^*)^2 = 4. \tag{4.9}$$

We can then parametrise the critical line by writing

$$X_0 = 2 \cosh u \quad X_1 = 2 \sinh u \tag{4.10}$$

so that on the critical line, $V[N, M]$ takes the form

$$\begin{aligned} V[N, M] = & \sum_{r, r'} N(r) \cdot e^{-ur_1} V_R(r - r') \cdot N(r') + 2i \sum_{r, \mathbf{R}} N(r) \cdot V_1(r - \mathbf{R}) M(\mathbf{R}) \\ & + \sum_{\mathbf{R}, \mathbf{R}'} M(\mathbf{R}) e^{u\mathbf{r}_1} V_R(\mathbf{R} - \mathbf{R}') M(\mathbf{R}'). \end{aligned} \tag{4.11}$$

A rather similar scheme holds for the eight-vertex model. In this case, $n_1(r)$ and $n_2(\mathbf{R})$ take on even integral values. Then define

$$\begin{aligned} N_1(r) &= n_1(r)/2 & N_2(r) &= m_2(r) \\ M_1(\mathbf{R}) &= m_1(\mathbf{R}) & M_2(\mathbf{R}) &= n_2(\mathbf{R})/2. \end{aligned} \tag{4.12}$$

From this definition and equations (3.31)–(3.33), we find

$$\begin{aligned} V[N, M] = & \frac{1}{2} \sum_{r, r'} N(r) \cdot V_R(r - r') \begin{bmatrix} \frac{4}{X_0} & i \frac{2X_1}{X_0} \\ i \frac{2X_1}{X_0} & \frac{X_0^2 - X_1^2}{X_0} \end{bmatrix} \cdot N(r') \\ & + 2i \sum_{r, \mathbf{R}} N(r) \cdot V_1(r - \mathbf{R}) M(\mathbf{R}) \\ & + \frac{1}{2} \sum_{\mathbf{R}, \mathbf{R}'} M(\mathbf{R}) \cdot V_R(\mathbf{R} - \mathbf{R}') \begin{bmatrix} \frac{X_0^2 - X_1^2}{X_0} & -i \frac{2X_1}{X_0} \\ -i \frac{2X_1}{X_0} & \frac{4}{X_0} \end{bmatrix} \cdot M(\mathbf{R}'). \end{aligned} \tag{4.13}$$

The self-duality and criticality condition for this model is once again equation (4.9). Once more the duality statement is simple representable by an interchange of N and M . In this critical domain, one can rewrite equation (4.13) in a form exactly parallel to that of equation (4.11) by using instead of (4.10) the parametrisation:

$$X_0 = \frac{2}{\cos \tilde{u}} \quad X_1 = 2 \tan \tilde{u} \tag{4.14}$$

which will once again ensure the satisfaction of equation (4.10). In this critical domain,

$$V[N, M] = \frac{1}{2} \sum_{\mathbf{r}, \mathbf{r}'} N(\mathbf{r}) V_{\mathbf{R}}(\mathbf{r} - \mathbf{r}') e^{-i\tilde{u}\tau_1} N(\mathbf{r}') \\ + 2i \sum_{\mathbf{r}, \mathbf{R}} N(\mathbf{r}) V_{\mathbf{I}}(\mathbf{r} - \mathbf{R}) M(\mathbf{R}) + \frac{1}{2} \sum_{\mathbf{R}, \mathbf{R}'} M(\mathbf{R}) V_{\mathbf{R}}(\mathbf{R} - \mathbf{R}') e^{-i\tilde{u}\tau_1} M(\mathbf{R}'). \tag{4.15}$$

4.3. Correlation functions for the Ising model

Equation (2.12a) shows that in the Ising model a non-zero magnetic field generates terms with odd n . If there were no magnetic field, then only terms with even n would appear. Thus observation translates into a simple rule for calculating a multiple spin correlation function. Let us represent the Ising model in the standard way with $\sigma(\mathbf{r}) = \pm 1$, being the possible values of the on-site variable. Then according to equation (2.12) a coupling

$$\sum_{\mathbf{r}} \sigma(\mathbf{r}) h(\mathbf{r})$$

translates into a vertex function

$$\sum_{\mathbf{r}} Y(n(\mathbf{r}), \mathbf{r})$$

with

$$e^{Y(h(\mathbf{r}), n(\mathbf{r}))} = \begin{cases} 2 \cosh h(\mathbf{r}) & \text{for } n(\mathbf{r}) \text{ even} \\ -2 \sinh h(\mathbf{r}) & \text{for } n(\mathbf{r}) \text{ odd.} \end{cases} \tag{4.16}$$

On one hand, we can calculate the multiple spin correlation function containing L spins as

$$C(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_L) = \left\langle \prod_{i=1}^L \sigma(\mathbf{r}_i) \right\rangle = \frac{(\pi_i \partial / \partial h(\mathbf{r}_i)) Z[h]}{Z[h]} \Big|_{h=0} \tag{4.17}$$

where $Z[h]$ is the partition function in the presence of a magnetic field. But, notice that equation (4.16) states

$$\frac{\partial}{\partial h} e^{Y(h, n)} \Big|_{h=0} = \begin{cases} 0 & \text{for } n \text{ even} \\ -1 & \text{for } n \text{ odd.} \end{cases}$$

Consequently, we find that

$$C(\mathbf{r}_1, \dots, \mathbf{r}_L) = (-1)^L Z' / Z. \tag{4.18}$$

Here Z is the partition function defined by equations (4.2) and (4.3) and Z' is the corresponding partition function defined with some of the N -values displaced by half-integral values. (Remember that n odd implies N half-integral for the Ising model.) In particular

$$Z' = \sum_{\{M,N\}} e^{A[N+\Delta,M]} \quad (4.19)$$

with

$$\Delta(\mathbf{r}) = \frac{1}{2} \sum_i \delta_{\mathbf{r}, \mathbf{r}_i} \quad (4.20)$$

This result can be generalised one step further. Recall that in the two-dimensional Ising model there exists a variable dual to $\sigma(\mathbf{r})$ called $\mu(\mathbf{R})$ (Kadanoff and Ceva 1971).

Define a correlation function involving $L - \sigma(\mathbf{r})$ variables and I dual $\mu(\mathbf{R})$ variables as

$$C(\mathbf{r}_1, \dots, \mathbf{r}_L; \mathbf{R}_1, \dots, \mathbf{R}_I) = \left\langle \left(\prod_{i=1}^L \sigma(\mathbf{r}_i) \right) \left(\prod_{j=1}^I \mu(\mathbf{R}_j) \right) \right\rangle = (-1)^L Z' / Z. \quad (4.21)$$

Because of the dual relation between N and M a natural definition of such a correlation function is to replace (4.18) by

$$Z' = \sum_{\{M,N\}} e^{A[N+\Delta, M+\Lambda]} \quad (4.22)$$

with

$$\Lambda(\mathbf{R}) = \frac{1}{2} \sum_j \delta_{\mathbf{R}, \mathbf{R}_j} \quad (4.23)$$

This definition of a $\sigma - \mu$ correlation function is precisely equivalent to the standard definition.

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